

## CROSS-COUPLING PLAYS CONSTRUCTIVE ROLE ON GLOBAL STABILITY OF SYNCHRONY IN NEURONAL NETWORKS

Suman Saha<sup>1,2</sup>, Arindam Mishra<sup>3</sup>, Prodyot Kumar Roy<sup>4\*</sup>, Syamal K. Dana<sup>2</sup>

<sup>1</sup> Department of Instrumentation and Electronics Engineering, Jadavpur University, Kolkata 700098, India;

<sup>2</sup> Department of Mathematics, Jadavpur University, Kolkata 700032, India;

<sup>3</sup> Department of Physics, Jadavpur University, Kolkata 700032, India;

<sup>4</sup> Department of Mathematics, Presidency University, Kolkata 700073, India.

\* Corresponding e-mail: pkpresi@yahoo.co.in

**Abstract.** Synchrony in neuronal networks plays a crucial role in the functioning of the brain. Stability of synchrony is most desirable to prevent any emergent desynchrony due to natural events, internal or external disturbances. The brain might have its own mechanism to repair its desynchrony, otherwise, some external procedure might be necessary to restore synchrony. We propose here a mechanism to realize robust synchrony in neuronal networks against parameter drifting. A selective addition of cross-coupling links over and above the conventional diffusive coupling links is found [Saha et al. (2017)] recently that makes dramatic improvements in the stability of synchrony of dynamical networks and that saves synchrony against breakdown due to parameter drifting. We apply the concept to realize globally stable synchrony in neuronal networks and the desired effect of robust synchrony and, present our numerical studies with examples of network motifs and a larger network of neurons and using the Hindmarsh-Rose (HR) [Hindmarsh and Rose (1984)] slow-fast neuron model for each node of the networks.

**Keywords:** Neuronal network, synchrony, cross-coupling and global stability, role of heterogeneity

### Introduction

The brain is a network of interacting neurons [Sporns (2010, 2013)] where their exact connectivity is not yet clearly known. A complete understanding of the brain structure and its function is still evading researchers; however, continuous research efforts are exerted in this direction. Synchrony of neurons plays important roles in the functioning of the brain [Singer (1993)]. Desynchronization in a neuronal network can hinder the desired performances of the brain that may emerge for many reasons, due to disease as internal perturbation or external effect. Some diseases are beyond control and permanent; in some other cases, the brain may have its own repair mechanism to restore synchrony in a situation of its breakdown. A disease may cause a temporary or permanent loss of a communication channel between the neurons or drift in system parameters. A large external disturbance as a temporary perturbation may also destroy the synchrony. A question naturally arises how such a loss of synchrony is to be taken care of and the network be prevented from the disaster. This is a very complex problem yet to have a clear answer.

We address the problem from a dynamical system viewpoint, although simplistic, that the brain is a dynamical network consisting of an ensemble of neurons; each neuron is represented by the simple slow-fast HR model. Synchrony in such a neuronal network is assumed to prevail with its known structure or its connecting links between the neurons. The basic concept of synchronization in oscillatory systems such as the neuron is briefly introduced here first; it is known [Pecora and Carroll (1998); Boccaletti et al. (2002); Pikovsky et al. (2003); Rosenblum et al. (1996)]

that a diffusive coupling or a mutually interactive link is always necessary between any two dynamical units of a network and when the strength of the coupling link is increased above a critical value, synchrony emerges in a network. If all the dynamical units are identical, the network emerges into complete synchrony (CS) [Pecora and Carroll (1998); Pikovsky et al. (2003)] when oscillations in all the units have identical amplitude and phase. In a network of neurons, they all oscillate in complete harmony of amplitude and phase and, thereby perform a desired task. Otherwise, for nonidentical units, and for a weaker coupling strength, a type of phase synchronization [Rosenblum et al. (1996); Roy et al. (2003)] may emerge when the dynamical units have only phase coherence, but no amplitude correlation. All the neurons in a network oscillate in phase coherence; amplitude of oscillations is only different. If the coupling strength is weakened further below a threshold, synchrony is lost between the neurons having no correlation either in amplitude or phase. Furthermore, if the system parameters are largely distributed, then also synchrony is lost. All the neurons oscillate in a complete random manner; even a cessation of oscillation [Mirollo and Strogatz (1990); Saxena et al. (2012)] may occur. Using this general understanding of synchrony in a dynamical network whose connectivity is assumed known a priori, we start with a synchronous network under the simplest diffusive coupling interactions, and target on how to prevent a loss of synchrony for a drifting of a parameter or appearance of a large external disturbance. We start with an ideal situation when all the neurons are identical. Our target is to ensure a global stability of synchrony in the neuronal network and realize its robustness against parameter perturbation.

The desirable effect is a prevention of desynchrony due to drifting of system parameters or a disturbance.

We must mention here that the conventional diffusive coupling can only ensure local stability of synchrony [Pecora and Carroll (1998)], which is restricted to a subbasin of the coupled system. As a result, the synchrony may break in a situation of large perturbation due to a sudden external flicker or disturbance; the synchronous state may move outside the subbasin and can never return to synchrony once the disturbance is over. On the other hand, if a global stability of synchrony is somehow established, it covers the whole basin of the coupled system and hence it remains robust against external large disturbances. It is found, in our previous study with many dynamical models [Saha et al. (2017); Padmanaban et al. (2015)] that addition of selective cross-coupling links over and above the diffusive coupling link can ensure global stability of synchrony in dynamical systems and its robustness against parameter drifting. We apply this strategy here for neuronal networks to achieve our desired target of global stability and other benefits. First, we take an example of a 2-node neuron model to illustrate numerically and analytically the conditions for realizing global stability of synchrony and to make it robust against drifting of system parameter and, then present numerical examples of 3-node, 4-node network motifs and a 16-node network to illustrate the constructive role of additional cross-coupling on synchrony. We specially choose the examples of network motifs since the network motifs are the building blocks of many real world networks, biological and physical [Milo et al. (2002)].

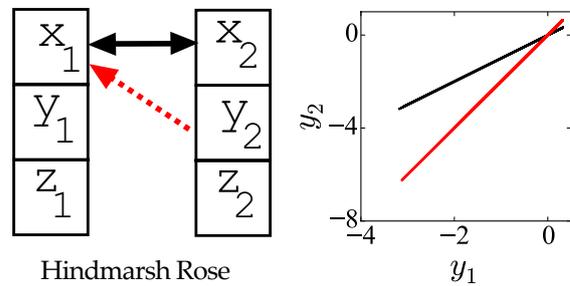
**Two coupled neurons**

Consider a simple 2-node HR system. As discussed above, addition of a selective cross-coupling link over and above the conventional self-diffusive coupling can make a dramatic improvement of synchronization. A systematic procedure is available [Saha et al. (2017)] to select the particular cross-coupling that can only help achieving our desired goal. We illustrate the main results analytically using the 2-coupled HR system as described by,

$$\begin{aligned}
 \dot{x}_1 &= ax_1^2 - x_1^3 + y_1 - z_1 + I + \varepsilon_1(x_2 - x_1) + \varepsilon_2(y_2 - y_1) \\
 \dot{x}_2 &= ax_2^2 - x_2^3 + y_2 - z_2 + I + \varepsilon_1(x_1 - x_2) \\
 \dot{y}_{1,2} &= b_{1,2}(1 - dx_{1,2}^2) - y_{1,2} \\
 \dot{z}_{1,2} &= \mu[r(x_{1,2} - c) - z_{1,2}]
 \end{aligned}
 \tag{1}$$

The coupling scheme is schematically shown in the left panel of Fig. 1. Each neuron is represented by three vertically stacked squares (left panel) and by their corresponding state variables ( $x_i, y_i, z_i$ ) where  $i = 1, 2$  corresponds to two neurons. The conventional bidirectional diffusive coupling link (solid black arrow) is represented by the coupling function  $(x_{2,1} - x_{1,2})$  and it is added to the  $x'_{1,2}$  dynamics of the coupled units. This

coupling function is called as a diffusive self-coupling whose strength is defined by  $\varepsilon_1$ . While a directed cross-coupling link (dashed red arrow),  $(y_2 - y_1)$ , is added only to the dynamics of  $x_1$ -variable of the first oscillator and its strength is  $\varepsilon_2$ . Note that the coupling function involves the  $y_{1,2}$  variables while adding to the  $x'$  dynamics. This is the reason why we call it as a cross-coupling by a comparison with the self-coupling. An alternative choice is possible, i.e.,  $(y_1 - y_2)$  might be added to the dynamics of  $x_2$ -variable of the second oscillator which does not affect the final results. Noteworthy that the choice of the cross-coupling is unique and very specific, which is chosen from the linear flow matrix of the HR system. The general principle of this choice of the coupling functions (both the self- and the cross-coupling functions) is explained, in detail, earlier [Saha et al. (2017)]. We elaborate here how the addition of the cross-coupling can dramatically improve the synchrony of the 2-coupled HR system.



**Figure 1.** Two coupled neurons (schematic diagram at left panel). Synchronization manifold (black line for CS and red line for GS) is projected in the  $y_1$ - $y_2$  plane at right panel. Parameters are chosen in the chaotic dynamical regions for both the systems:  $a = 3, I = 3.25, d = 5, \mu = 0.006, r = 4, c = 1.6, b_1 = 0.5, b_2 = 1. \varepsilon_1 = 0.45$  and  $\varepsilon_2 = 1$ .

The selected coupling profile (diffusive self-coupling and cross-coupling functions) favors global stability of synchrony in the two-coupled neuron model. Analytical details are provided in the Appendix to derive the condition for the coupling strengths,  $\varepsilon_1 \leq a^2/6$  and  $\varepsilon_2 = 1$ , when a globally stable complete synchronization (CS) state i.e.,  $x_1 = x_2, y_1 = y_2$  and  $z_1 = z_2$  emerges for the coupled neurons. This condition explains that both the neurons oscillate in complete coherence of amplitude and phase. The amplitudes and phases of the neurons remain same all the time. The synchronous state is clearly globally stable as proved in the Appendix; it is stable for any choice of initial conditions from the basin of attraction of the coupled system. Numerical results are presented in the right panel of Fig. 1 for two identical neurons. The 2D projection of the synchronization manifold of identical neurons is shown in a black line. The red line also confirms a synchronous state, but it is a type of generalized synchronization (GS) [Abarbanel et al. (1996)] that emerges for a parameter perturbation. This is also globally stable which is never possible under a simple diffusive self-coupling. For illustration, we introduce a parameter mismatch i.e.,  $(b_1 \neq b_2)$ , when the globally stable GS state defined by  $y_1 = \lambda y_2$  and  $z_1 = \lambda z_2$  emerges (see Appendix). We note that  $x_{1,2}$  and  $z_{1,2}$  variables achieve an identical state similar to the CS case, however,  $y_1$  and  $y_2$

variables are not identical, but develop a linear relation where the linearity constant depends upon the ratio of the system parameters of the neurons. In the right panel of Fig. 1, a 2D projection of the GS manifold (red line) of the detuned systems shows a rotation from the CS manifold. A transition from CS to GS state simply occurs due to the parameter perturbation. This indicates that if a parameter of one neuron is drifted or changed, the CS state simply transits to the GS state without a loss of overall coherence or synchrony. One of the state variables of the perturbed neuron is either amplified or attenuated depending upon the value of the detuned parameter. The GS thus can be viewed as an amplitude response of the detuned system's attractor [Saha et al. (2017)] under parameter perturbation. This phenomenon is further explained in the next section.

**Network of neurons**

We now extend our results to some examples of network motifs. Consider a network of N-oscillators, whose dynamics of the i-th node under both self- diffusive and cross-coupling is governed by,

$$\begin{aligned} \dot{x}_i &= ax_i^2 - x_i^3 + y_i - z_i + I + \varepsilon_1 \sum_{j=1}^N A_{ij}(x_j - x_i) \\ &+ \varepsilon_2 \sum_{j=1}^N B_{ij}(y_j - y_i) \\ \dot{y}_i &= b_i(1 - dx_i^2) - y_i \\ \dot{z}_i &= \mu[r(x_i - c) - z_i] \end{aligned} \tag{2}$$

$A=\{A_{ij}\} \in R^{N \times N}$  is the adjacency matrix that defines the topology or structure of any network, in general, via the self-coupling links.

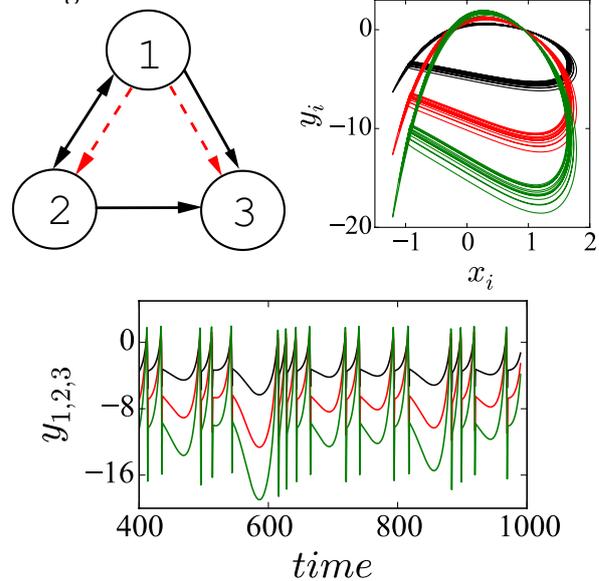
$B=\{B_{ij}\} \in R^{N \times N}$  is the connectivity matrix of the additional cross coupling links of the network;  $A_{ij}=1$  and  $B_{ij}=1$ , if i-th node is connected to the j-th ( $j \neq i$ ) and 0 otherwise. B is selected from A: the directed cross-coupling links are connected from the maximum outdegree node to all the other nodes so that the network structure remains unaltered.  $\varepsilon_1$  and  $\varepsilon_2$  are the strength of self- and cross-coupling links, respectively, between any two nodes whose values are analytically derived from Lyapunov stability of synchrony.

$\varepsilon_1 = 1.9$  and  $\varepsilon_2 = 1$  are taken for the 3-, 4-node motifs and the 16-node network.

We consider a 3-node motif first as shown in the upper left panel of Fig. 2 with three self-coupling links (black arrows), two directed links and one bidirectional link. The addition of two directed cross-coupling links (red dashed arrows) is suggested by our proposed strategy [Saha et al. (2017)] that makes synchronization globally stable and robust to parameter perturbation. Accordingly, we change  $A_{ij}$  and  $B_{ij}$  in Eq. (2) to define the dynamics of the 3-node motif. In upper right panel, the 2D projection of the attractors of three nodes are drawn for different values of  $b_1 = 1, b_2 = 2$  and  $b_3 = 3$  in black, red and green, respectively, which depicts increasing enlargement of the attractors along the  $y_i$  direction. The enlargement increases with positive detuning of parameters in the nodes (node 2 and 3). For three identical nodes, all the attractors collapse on the black attractor

when a globally stable CS in the motif is established (analytical proof is not presented here). For all three different nodes, an overall coherence is maintained between the three nodes; three different attractors (black, red and green) with different amplification constant are seen for positive detuning of the parameters. This state is defined as the GS state and it is also globally stable. Thus parameter mismatches in three nodes do not break the synchrony. Lower panel shows the corresponding  $y_i$  time series of three nodes where all the oscillations are perfectly in phase, but different in amplitude depending upon the value of the detuned parameter of a particular node. Note that other state variables  $x_{1,2,3}$  and  $z_{1,2,3}$  remain identical similar to the example of two coupled neurons given in the previous section. Results support our desired target that the 3-node motif is globally stable in a CS state by the addition of the cross-coupling links when they are all identical. For all perturbed nodes, represented by their different parameters, the motif transits to a GS state which is also globally stable; there is no loss of overall synchrony.

The adjacency matrix A and the connectivity matrix B are given for the 3-node



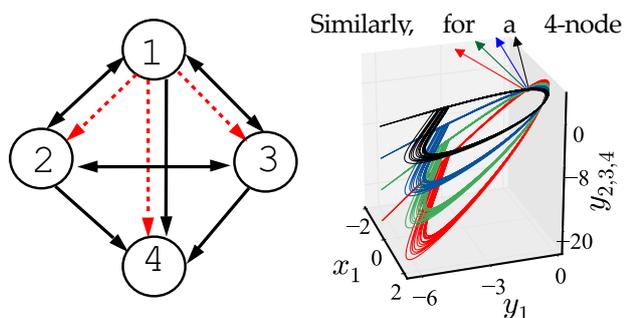
**Figure 2.** A 3-node network motif (upper left panel). Black arrows for self-coupling links and red arrows for cross-coupling links. Upper right panel shows 2D phase portraits in the  $x_i$ - $y_i$  plane of three neurons for  $b_1=1, b_2=2$  and  $b_3=3$  in black, red and green, respectively. Lower panel shows  $y_i$  time series of three neurons.

motif,

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}; B = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

and for the selected 4-node motif,

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix}; B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$



**Figure 3.** A 4-node network motif (left panel). Solid (black) and dashed (red) arrows represent self- and cross-coupling links, respectively. Right panel: projection of synchronization hyperplanes. Color arrows show rotation along the transverse direction to their respective hyperplanes. Parameters are  $b_1=1$ ,  $b_2=2$ ,  $b_3=3$ ,  $b_4=4$ , respectively, for black, blue, green and red hyperplanes.  $\epsilon_1 = 1.9$  and  $\epsilon_2 = 1$ .

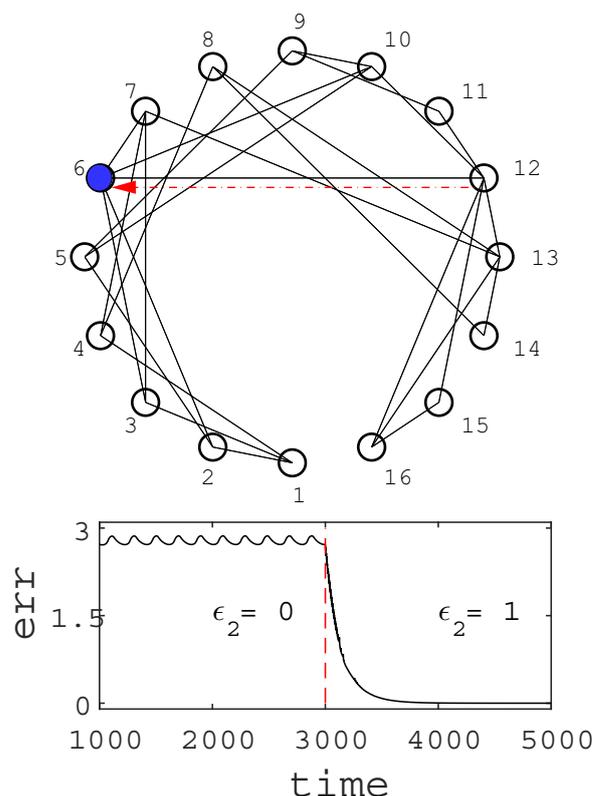
motif shown in the left panel of Fig. 3, we follow the same strategy of applying the cross-coupling links (red dashed arrows). From the beginning, we assume that all the nodes are perturbed by their parameters ( $i=1, 2, 3, 4$ ). On the right panel, the synchronization hyperplanes of the detuned oscillators (blue, green and red) against the identical state (black line) are presented. A comparison clearly shows that the synchronized attractors are gradually enlarged with increasing value of the parameter. The 4-node motif attains a GS state that remains globally stable. The color arrows indicate the transverse direction to the synchronization hyperplanes, which clearly shows their rotational effect with increasing parameter mismatch. We repeat that all the nodes collapse to the synchronization hyperplane in black when they are identical and attain a globally stable CS state, as described above for previous two examples.

The effect of globally stable synchrony and its robustness to parameter perturbation is also noticed in a larger network with 16-node as shown in the upper panel of Fig. 4. The original network is defined by its nodes and bidirectional self-coupling (black links) which is assumed synchronous for a coupling strength  $\epsilon_1 (=1.9)$  larger than a critical value. However, if we detune the parameter of any one of the nodes, say,  $b_6$  of node-6 (blue node), with respect to the other nodes, the synchrony in the network is lost. By adding one directed cross coupling link (red dashed arrow) from any of its neighbors (say, from node-12 to node-6), the synchrony is restored, which is shown in the lower panel of Fig. 4. Synchronization in the network is monitored by defining an error,

$$err(t) = \sqrt{\sum_{i=2}^N [(x_1 - x_i)^2 + (z_1 - z_i)^2]}$$

where  $N$  is the network size.  $err(t) \neq 0$  means no synchronization. The synchronization error ( $err$ ) is large before adding the cross-coupling link ( $\epsilon_2 = 0$ ) and becomes 0 when the cross-coupling link is added ( $\epsilon_2 = 1$ ). Addition of a cross-coupling link restores synchrony immediately.

We are yet to prove its global stability. In fact, a transition from the original CS state (before addition of the cross-coupling link) to a GS state (after the addition) is also seen, but we do not elaborate it here. We present here only the constructive role of a cross-coupling link that is clear from our numerical result.



**Figure 4.** Upper panel shows schematic diagram of a network. Node-6 (blue node) is perturbed while all other nodes have identical parameters. To restore synchrony only one directed cross-coupling link (dashed red arrow) is added to node-6 from a neighbor node-12. In the lower panel, time series of the synchronization error ( $err$ ) shows that in absence of the cross-coupling link ( $\epsilon_2 = 0$ ), synchrony is lost when node-6 is perturbed ( $b \neq b_6 = 2$ ). At  $t \geq 3000$ , one directed cross-coupling link ( $\epsilon_2 = 1$ ) is added from node-12 to node-6, synchrony is restored ( $err = 0$ ) in the network.

### Conclusion

We exemplified here the constructive role of selective addition of cross-coupling links over and above the conventional diffusive coupling links in neuronal networks to improve the conditions of synchrony so that it can sustain under external disturbances, large perturbations or parameter drift with time.

We proposed that establishing a global stability of synchronization can save a neuronal network from a breakdown of synchrony, to an extent, from large external disturbance. The addition of cross-coupling links as tested earlier for quite a few network motifs of many dynamical models helped achieve the global stability of synchrony in the network and furthermore, maintain an overall coherence among all the nodes when a parameter of any one or more nodes is perturbed.

We provided analytical and numerical proof in favor of our propositions in a 2-node coupled neurons, numerical examples of a 3-node, a 4-node network motifs and a 16-node random network. We showed how addition of cross-coupling links realized global stability of synchrony and its robustness against parameter perturbation. Each node of the network is represented by the HR neuron model. The global stability of synchrony was maintained for identical nodes as well as nonidentical nodes. The synchrony was complete with identical amplitude and phase for the identical case; this

CS state transited to a GS state for nonidentical nodes, in other words, parametrically perturbed nodes. One or more state variables of the perturbed nodes showed amplitude response (either amplified or attenuated depending upon the amount of detuning from the other nodes). We must mention that this is a very simplistic representation of the real problem in a neuronal network. We use the dynamical system approach to propose a purely theoretical solution to address the problem.

**Acknowledgements**

S.S is supported by the SERB-DST (India), A.M is supported by the University Grants Commission (India). P.K.R. acknowledges support by the CSIR (India) Emeritus scientist scheme and S.K.D. is supported by the University Grants Commission (India) Emeritus Fellowship.

**Appendix:** Global stability analysis of synchronization for two nodes

The error functions

$$e = [e_x, e_y, e_z]^T = [x_1 - x_2, y_1 - y_2, z_1 - z_2]$$

of the systems Eq.(1) evolves as,

$$\begin{aligned} \dot{e}_x &= ae_x e_p - \frac{e_x^3}{4} - \frac{3}{4} e_x e_p^2 + e_y - e_z - 2\varepsilon_1 e_x - \varepsilon_2 e_y \\ \dot{e}_y &= b_1(1 - dx_1^2) - b_2(1 - dx_2^2) - e_y \\ \dot{e}_z &= \mu r e_x - \mu e_z, \end{aligned} \tag{3}$$

Where,  $e_p = x_1 + x_2$  so that  $x_1^2 - x_2^2 = e_x e_p$  and  $x_1^3 - x_2^3 = \frac{e_x}{4}(e_x^2 + 3e_p^2)$ .

For a global stability of  $(e_x = 0, e_y = 0, e_z = 0)$ , we consider a Lyapunov function,  $V(e) = \frac{1}{2}e_x^2 + \frac{1}{2}e_y^2 + \frac{1}{2}e_z^2$ . We first check the stability of  $x_1 = x_2$  and  $z_1 = z_2$ , separately, by defining a

Lyapunov function,  $V'(e_x, e_z) = \frac{1}{2}e_x^2 + \frac{1}{2\mu r}e_z^2$

when its time derivative is

$$\begin{aligned} V'(e_x, e_y) &= -e_x^2 \left( \frac{3}{4}e_p^2 - ae_p + 2\varepsilon_1 \right) - \frac{e_x^4}{4} \\ &\quad + (1 - \varepsilon_2)e_y e_x - \frac{e_z^2}{r} \end{aligned} \tag{4}$$

$V'(e_x, e_z) < 0$  provided  $\varepsilon_2 = 1$  and

$$\frac{3}{4}e_p^2 - ae_p + 2\varepsilon_1 \geq 0.$$

To satisfy the condition, we derive the roots of the equation,

$$\frac{3}{4}e_p^2 - ae_p + 2\varepsilon_1 = 0 \tag{5}$$

which are given by

$$e_{p_{1,2}} = \frac{2a \pm \sqrt{a^2 - 6\varepsilon_1}}{3} \tag{6}$$

$e_{p_{1,2}}$  will now be positive real if  $a^2 - 6\varepsilon_1 \geq 0$  that implies  $\varepsilon_1 \leq \frac{a^2}{6}$ . Thus  $V'(e_x, e_z) < 0$  is satisfied

when  $\varepsilon_2 = 1$  and  $\varepsilon_1 \leq a^2 / 6$  and this implies  $x_1 = x_2$  and  $z_1 = z_2$  is asymptotically stable as  $t \rightarrow \infty$ .

Substituting the condition in Eq.(3) and assuming identical systems ( $b_1 = b_2$ ), the error dynamics  $\dot{e}_y = -e_y$  is found when,

$$\dot{V}(e_x, e_y, e_z) = -\frac{e_x^4}{4} - e_y^2 - \frac{e_z^2}{r} < 0 \tag{7}$$

for  $\varepsilon_1 \leq a^2 / 6$  and  $\varepsilon_2 = 1$ . Under this condition, the CS state  $x_1 = x_2, y_1 = y_2$  and  $z_1 = z_2$  becomes globally stable.

Now we consider the effect of heterogeneity on the stability of CS state by detuning the parameter ( $b_1 \neq b_2$ ). As the induced heterogeneity is not effecting the Eq.(4) and the related condition, hence the stability, hence the stability of  $x_1 = x_2$  and  $z_1 = z_2$  is still preserved. After detuning, the equation of  $\dot{e}_y$  from Eq. (3) becomes,

$$\begin{aligned} \dot{e}_y &= b_1(1 - dx_1^2) - b_2(1 - dx_2^2) - e_y \\ &= (b_1 - b_2)(1 - dx_1^2) - e_y \\ &= \frac{b_1 - b_2}{b_1}(\dot{y}_1 + y_1) - e_y \end{aligned} \tag{8}$$

and this leads to

$$\begin{aligned} \dot{y}_1 \left( 1 - \frac{b_1 - b_2}{b_1} \right) - \dot{y}_2 &= -y_1 \left( 1 - \frac{b_1 - b_2}{b_1} \right) + y_2 \\ \dot{y}_1 \frac{b_1}{b_1} - \dot{y}_2 &= -y_1 \frac{b_1}{b_1} + y_2. \end{aligned} \tag{9}$$

From (9), the received error dynamics is  $\dot{e}_y^* = -e_y^*$ ,

where  $e_y^* = y_1 \frac{b_1}{b_1} - y_2$

is the modified error function. Accordingly, the Lyapunov function is redefined in terms of the modified error function whose time derivative is

$$\dot{V}(e_x, e_y^*, e_z) = -\frac{e_x^4}{4} - e_y^{*2} - \frac{e_z^2}{r} < 0 \tag{10}$$

which ensures globally stable synchronization.

**References**

Abarbanel, H. D., Rulkov, N. F., and Sushchik, M. M. (1996). Generalized synchronization of chaos: The auxiliary system approach. *Physical Review E*, 53(5):4528.

Boccaletti, S., Kurths, J., Osipov, G., Valladares, D., and Zhou, C. (2002). The synchronization of chaotic systems. *Physics reports*, 366(1):1-101.

Hindmarsh, J. L. and Rose, R. (1984). A model of neuronal bursting using three coupled first order differential equations. *Proceedings of the royal society of London B: biological sciences*, 221(1222):87-102.

Milo, R., Shen-Orr, S., Itzkovitz, S., Kashtan, N., Chklovskii, D., and Alon, U. (2002). Network motifs: simple building blocks of complex networks. *Science*, 298(5594):824-827.

- Mirollo, R. E. and Strogatz, S. H. (1990). Amplitude death in an array of limit-cycle oscillators. *Journal of Statistical Physics*, 60(1):245–262.
- Padmanaban, E., Saha, S., Vigneshwaran, M., and Dana, S. K. (2015). Amplified response in coupled chaotic oscillators by induced heterogeneity. *Phys. Rev. E*, 92:062916.
- Pecora, L. M. and Carroll, T. L. (1998). Master stability functions for synchronized coupled systems. *Physical review letters*, 80(10):2109.
- Pikovsky, A., Rosenblum, M., and Kurths, J. (2003). *Synchronization: a universal concept in nonlinear sciences*, volume 12. Cambridge university press.
- Rosenblum, M. G., Pikovsky, A. S., and Kurths, J. (1996). Phase synchronization of chaotic oscillators. *Physical review letters*, 76(11):1804.
- Roy, P. K., Chakraborty, S., and Dana, S. K. (2003). Experimental observation on the effect of coupling on different synchronization phenomena in coupled nonidentical chua's oscillators. *Chaos: An interdisciplinary Journal of Nonlinear Science*, 13(1):342–355.
- Saha, S., Mishra, A., Padmanaban, E., Bhowmick, S. K., Roy, P. K., Dam, B., and Dana, S. K. (2017). Coupling conditions for globally stable and robust synchrony of chaotic systems. *Physical Review E*, 95(6):062204.
- Saxena, G., Prasad, A., and Ramaswamy, R. (2012). Amplitude death: The emergence of stationarity in coupled nonlinear systems. *Physics Reports*, 521(5):205–228.
- Singer, W. (1993). Synchronization of cortical activity and its putative role in information processing and learning. *Annual Review of Physiology*, 55(1):349–374. PMID: 8466179.
- Sporns, O. (2010). *Networks of the Brain*. MIT press.
- Sporns, O. (2013). Structure and function of complex brain networks. *Dialogues in clinical neuroscience*, 15(3):247